D - Optimum Parameters Estimation Of Models Of Stochastic Linear Discrete- Time Systems In Frequency Domain

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Abstract-The procedure of optimum parametrical estimation, taking into account various levels of parametrical apriori uncertainty is presented. The results are given for the case when the parameters estimated are included in various combinations in the state, control, noise and measurement matrices, initial conditions and covariance matrices of dynamic noise and measurement errors.

I. INTRODUCTION

The problem of active identification (optimum estimation) is one of the basic problems of the theory and practice of automatic control and consists in construction of mathematical model from experimental observation of an input and output of dynamic system.

The active identification at the expense of using the most informative observational data allows to raise accuracy of estimation of the unknown parameters. In this case some functional of information or variance matrix of vector of estimated parameters can be used as measure of informativeness.

The procedure of active identification consists of four stages. The first stage assumes calculation of estimations of parameters from observational data, appropriate to some trial signal (passive identification). These estimations are used at the second stage for synthesis of optimal signal by some criterion (planning). The third stage consists in registration of output data of the system corresponding to the sent synthesized signal. The received observational data are used at the fourth stage for recalculation of estimations of unknown parameters. In case of necessity the stages 2-4 repeat.

The task of active identification of stochastic linear discrete-time systems in time-domain area for the most general case of entry of unknown parameters was considered and stated in detail in [1]. The idea of application of the frequency approach to the decision of a considered task was offered by R.K. Mehra in [2,3] and further was advanced by A.Zh. Abdenov in [4]. Thus the estimated parameters were included into state and control matrices.

Let's present the developed procedures of optimal estimation, combining methods of traditional parametrical estimation with original algorithms of planning of input signals for a case, when estimated parameters are not only in models of dynamics and observation, but also enter into the initial conditions, and also in variance matrix of noise of dynamics and observational errors.

II. STATEMENT OF THE PROBLEM

Let's consider the following stationary identifiable model of controllable, observable linear discrete - time system:

$$x(t+1) = \Phi x(t) + \Psi u(t) + \Gamma w(t), \quad (1)$$

$$y(t+1) = Hx(t+1) + v(t+1), \; t = 0,1,\ldots,N-1. \quad (2)$$

Here $x(t)$ - n-vector of a condition; $u(t)$ - determined r-vector of control (input); $w(t)$ - p-vector of noise; $y(t+1)$ - m-vector of observation (output), $v(t+1)$ - m-vector of an error of observation, Φ, Ψ, Γ, H - matrixes of the appropriate dimensions. The random vectors $w(t)$ and $v(t+1)$ are stationary white Gaussian sequences, and

$$E[w(t)] = 0, \; E[w(t)w^T(\tau)] = Q\delta_{t,\tau}; \; E[v(t+1)] = 0,$$

$$E[v(t+1)v^T(\tau+1)] = R\delta_{t,\tau}, \; E[v(t)w^T(\tau)] = 0;$$

for any $t, \tau = 0,1,\ldots,N-1$. The initial condition $x(0)$ has normal distribution with parameters $\bar{x}(0)$, $P(0)$, and does not correlate with $w(t)$ and $v(t+1)$ for any values of variable t.

It is necessary to obtain in frequency domain optimal estimates of unknown parameters $\Theta = (\theta_1, \theta_2, \ldots, \theta_s)$ which are included in various combinations in matrices $\Phi, \Psi, \Gamma, H, Q, R, P(0)$ and a vector $\bar{x}(0)$.

III. PARAMETERS ESTIMATION

We shall present all time domain sequences which are included in model (1) - (2) in the terms of discrete transformation Furie, having used the following relation [5]:

$$\tilde{a}(k) = \frac{1}{N}\sum_{t=0}^{N-1} a(t) e^{\frac{2\pi i k t}{N}} = \frac{1}{N}\sum_{t=0}^{N-1} a(t) z_k^t, \; z_k = e^{\frac{2\pi k i}{N}},$$

$$k = 0, 1, \ldots, N.$$

In result it is possible to receive analogue to the model (1) - (2):

$$z_k\tilde{x}(k) = \Phi\tilde{x}(k) + \tilde{K}\tilde{\varepsilon}(k) + \Psi\tilde{u}(k)\,, \qquad (3)$$

$$\tilde{y}(k) = H\tilde{x}(k) + \tilde{\varepsilon}(k)\,,\ k=0,1,\ldots,N, \qquad (4)$$

where $\tilde{\varepsilon}(k)$ и $\tilde{K}$ are calculated from the following equations of the Kalman filter, corresponding to the established mode of the system:

$$\tilde{x}(k) = (z_k I - \Phi)^{-1}\Psi\tilde{u}(k)\,;$$

$$\tilde{\varepsilon}(k) = \tilde{y}(k) - H\tilde{x}(k)\,;$$

$$B = HPH^T + R;$$

$$\tilde{K} = \Phi PH^T B^{-1}\,,$$

and the matrix P is calculated by the decision of the discrete Riccati equation [6]:

$$-P + \Phi P\Phi^T - \Phi PH^T(HPH^T + R)^{-1}HP\Phi^T + \Gamma Q\Gamma^T = 0$$

For calculation of estimates of unknown parameters we shall use methods of the maximum likelihood:

$$\hat{\Theta}_{ML} = \arg\min_{\theta\in\Omega_\Theta}\{-\ln L(\tilde{Y};\Theta)\}\,, \qquad (5)$$

where

$$\ln L(\tilde{Y};\Theta) = -\frac{Nm}{2}\ln 2\pi - \frac{N}{2}\ln\det B - \frac{N}{2}\sum_{k=0}^{N-1}\tilde{\varepsilon}^*(k)B^{-1}\tilde{\varepsilon}(k)\,, \qquad (6)$$

$$\tilde{Y} = \{\tilde{y}^T(1),\ldots\tilde{y}^T(N)\}.$$

The decision of an extreme task (5) is carried out by a method of Newton -Gausse.

IV. DESIGN OF INPUT SIGNALS

By continuous normalized plan we will understand

$$\xi = \begin{Bmatrix}\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_q \\ p_1,\ p_2, \ldots,\ p_q\end{Bmatrix},\ \sum_{i=1}^{q} p_i = 1\,,\ p_i \ge 0\,,$$

$$\tilde{U}_i \in \Omega_{\tilde{U}}\,,\ i=1,2,\ldots,q \qquad (7)$$

The control by signal on the basis of methods of experimental design assumes presence of an information matrix of the plan, which is used in algorithms of numerical construction of the optimum designs.

The information matrix of the plan (7) is connected to information matrices $M(\tilde{U}_i)$ of points of a spectrum of the plan by relation:

$$M(\varepsilon) = \sum_{i=1}^{q} p_i M(\tilde{U}_i),$$

where

$$M(\tilde{U}) = -\mathop{E}_{\tilde{Y}}\left[\frac{\partial^2 \ln L(\tilde{Y};\Theta)}{\partial\theta\partial\theta^T}\right].$$

The likelihood function is calculated with the help of (6).

For model (3), (4) elements of an information matrix of the design concentrated in one point are calculated by the formulas [7]:

$$M_{ij}(\tilde{U};\theta) = N\sum_{k=0}^{N-1}\mathrm{Tr}\left(A(z_k,\theta)\tilde{u}(k)\tilde{u}^*(k)\right) + \sum_{k=0}^{N-1}\mathrm{Tr}B(z_k,\theta) + \frac{N}{2}\sum_{k=0}^{N-1}\mathrm{Tr}\left[B^{-1}\frac{\partial B}{\partial\theta_i}B^{-1}\frac{\partial B}{\partial\theta_j}\right]\Bigg\}\,,\ i,j=\overline{1,s}\,,$$

where

$$A(z_k,\theta) = \frac{\partial T_1^*}{\partial\theta_i}\left(T_2^*\right)^{-1}B^{-1}T_2^{-1}\frac{\partial T_1}{\partial\theta_j}\,,$$

$$B(z_k,\theta) = T_2^{-1}\frac{\partial T_2}{\partial\theta_j}\frac{\partial T_2^*}{\partial\theta_i}\left(T_2^{-1}\right)^* + T_2^{-1}\frac{\partial T_2}{\partial\theta_i}T_2^{-1}\frac{\partial T_2}{\partial\theta_j} + T_2^{-1}\frac{\partial T_2}{\partial\theta_j}T_2^{-1}\frac{\partial T_2}{\partial\theta_i} - T_2^{-1}\frac{\partial^2 T_2}{\partial\theta_i\partial\theta_j} + T_2^{-1}\frac{\partial T_2}{\partial\theta_j}\frac{\partial B}{\partial\theta_i}B^{-1} + T_2^{-1}\frac{\partial T_2}{\partial\theta_i}\frac{\partial B}{\partial\theta_j}B^{-1}\,,$$

$$T_1(z_k,\theta) = H(z_k I - \Phi)^{-1}\Psi\,,$$

$$T_2(z_k,\theta) = H(z_k I - \Phi)^{-1}\tilde{K} + I$$

In the theory of experimental design a lot of criteria of optimality directed on the increasing to accuracy of estimation of unknown parameters is used. One of most widespread criterion is the criterion of D- optimality.

The plans ξ^*, which are optimum by this criterion, satisfy to a condition:

$$\xi^* = \arg\max_{\xi\in\Omega_\xi}\det M(\xi)\ . \qquad (8)$$

For D-optimum designing volume of ellipsoide of dispersion of estimations of the parameters is minimized.

The decision optimization task (8) is carried out directly through a method of gradient projection with application of reccurent algorithms of calculation of gradients of the following functionals:

$$\nabla_{\tilde{U}}(-\ln\det[M(\xi)]) = \left\|\frac{\partial(-\ln\det[M(\xi)])}{\partial\tilde{u}_j^{(i)}(k)}\right\|;$$

$$i = 1,\ldots,q;\ j = 1,\ldots,r;\ k = 0,\ldots,N-1\,,$$

and

$$\nabla_{\tilde{P}}(-\ln\det[M(\xi)]) = \left\|\frac{\partial(-\ln\det[M(\xi)])}{\partial p_i}\right\|;\ i = 1,\ldots,q$$

V. EXAMPLE

Example of mathematical model of system of (1), (2) is following discrete-time model of change of temperature in a two-room apartment with adjacent rooms and heater which is turning out from determined continuous model of dynamics at temperature capacity [8] of air 1 Kdj/kg and weights of air 64 kg and 25 kg in each of rooms accordingly:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \Psi u(t) + \begin{pmatrix} 0.313 & 0 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix};$$

$$y(t+1) = x_1(t+1) + v(t+1), \ t=0,1,...,N-1,$$

where the elements of matrixes $\|\Phi\|_{ij} = \varphi_{ij}$, $\|\Psi\|_{ij} = \psi_{ij}$ determined by the relations:

$$\varphi_{11} = -\frac{1}{2} e^{\frac{1}{2}\sqrt{(\alpha+\beta)\beta}} - e^{\frac{1}{2}\sqrt{(\alpha+\beta)}} + a e^{\frac{1}{2}\sqrt{(\alpha-\beta)}} - a e^{\frac{1}{2}\sqrt{(\alpha+\beta)}} + c e^{\frac{1}{2}\sqrt{(\alpha+\beta)}} - c e^{\frac{1}{2}\sqrt{(\alpha-\beta)}},$$

$$\varphi_{12} = \varphi_{21} = -e^{\frac{1}{2}\sqrt{(\alpha-\beta)}} - \frac{b}{\beta} e^{\frac{1}{2}\sqrt{(\alpha+\beta)}},$$

$$\varphi_{22} = \frac{1}{2} e^{\frac{1}{2}\sqrt{(\alpha+\beta)\beta}} + e^{\frac{1}{2}\sqrt{(\alpha+\beta)}} + a e^{\frac{1}{2}\sqrt{(\alpha-\beta)}} - a e^{\frac{1}{2}\sqrt{(\alpha+\beta)}} + c e^{\frac{1}{2}\sqrt{(\alpha+\beta)}} - c e^{\frac{1}{2}\sqrt{(\alpha-\beta)}};$$

$$\psi_{11} = 2c^2\beta\gamma e^{-\frac{1}{2}\beta} - ac\gamma e^{-\frac{1}{2}\beta} + c^2\gamma e^{-\frac{1}{2}\beta} + 2b^2\gamma e^{-\frac{1}{2}\beta} - c^2\beta\gamma e^{\frac{1}{2}\beta} - ac\gamma e^{\frac{1}{2}\beta} + \frac{2}{\alpha^2-\beta^2},$$

$$\psi_{22} = 2abc\gamma e^{\frac{1}{2}\beta} + ac\gamma e^{\frac{1}{2}\beta} + \beta\gamma e^{\frac{1}{2}\beta} - a\gamma e^{-\frac{1}{2}\beta} - c\gamma e^{-\frac{1}{2}\beta} + \beta\gamma e^{-\frac{1}{2}\beta} - \frac{2}{\alpha^2-\beta^2};$$

and

$$\alpha = -0.056\theta_1 - 2\theta_2, \ \beta = \sqrt{0.001\theta_1 + 4\theta_2^2}), \ \gamma = e^{0.012\theta_1};$$

Let's choose N=18, $\bar{x}(0) = (20, 20)$, R=0.1, $Q = P(0) = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$.

Let's consider that the control of quantity of heat selected by a heater changes from 20 Kdj/min up to 80 Kdj/min:

$$\Omega_U = \{ U \in R^N \,|\, 20 \le u(t) \le 80, \ t = 0,1,\ldots 17 \}.$$

Let's apply the procedure of active identification for an estimation of unknown parameters θ_1, θ_2 included in matrices of state and control. For evident representation of the received frequency results we shall transform them to time area using inverse Furie transformation [5].

Choosing an initial input signal according to a fig. 1

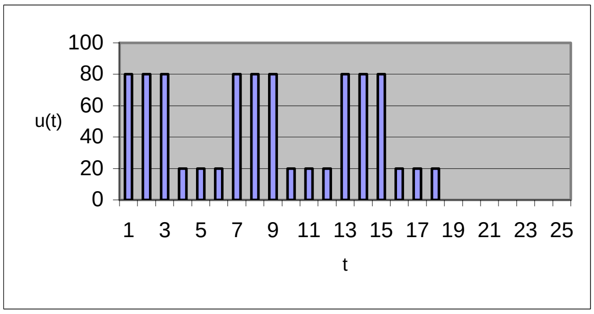

Fig. 1. Initial input signal u(t)

Let's receive estimations of unknown parameters on zero iteration $\hat{\theta}_1^0$ =0.26 and $\hat{\theta}_2^0$ =0.68, while true values of parameters are $\hat{\theta}_1 = 0.8$, $\hat{\theta}_2 = 0.1$, thus $\ln\det M(\hat{\theta}^0)$ = = 156.2 and evclide norm of a difference of vectors of observations and estimations of one-step-by-step forecasting is $\| \tilde{Y}^0 - \hat{Y}^0 \|$=3.859.

By these estimations on the following iteration the optimum input signal is synthesized, on the basis of which we recalculate estimations of parameters. Then we shall repeat process as a result to complete the procedure of optimum estimation nine iterations were needed.

The input signal appropriate to last iteration is submitted in a fig. 2.

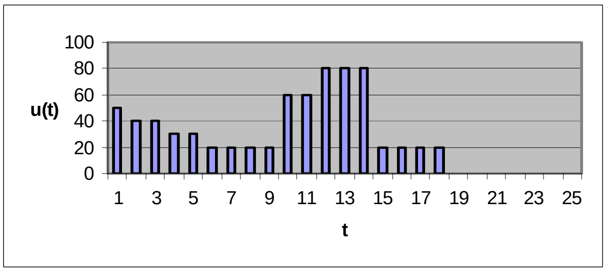

Fig. 2. Synthesized signal u(t)

The estimations appropriate to this input signal of unknown parameters are $\hat{\theta}_1^9$ =0.559 и $\hat{\theta}_2^9$ =0.312, and $\ln\det M(\hat{\theta}^9)$ =195.2, $\| \tilde{Y}^9 - \hat{Y}^9 \|$=1.608.

Thus in space of parameters it was possible to improve an estimation on $\frac{\sqrt{\|\hat{\theta} - \hat{\theta}^9\|}}{\sqrt{\|\hat{\theta} - \hat{\theta}^0\|}} * 100\% \approx 24\%$.

For comparison of results in space of the responses we shallpresent in a fig. 3-6 diagrams of dependence of real and imaginary parts of vectors of observation and errors of one-step-by-step forecasting on zero and ninth iteration.

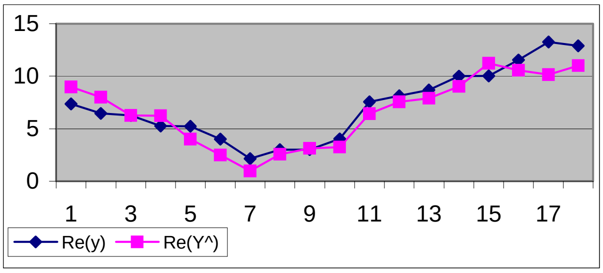

Fig. 3. Dependence of real parts of vectors of observation and errors of one-step-by-step forecasting on zero iteration

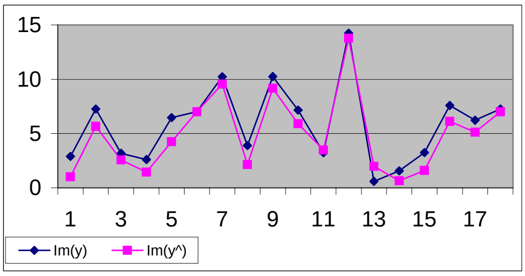

Fig. 4. Dependence of imaginary parts of vectors of observation and errors of one-step-by-step forecasting and zero iteration

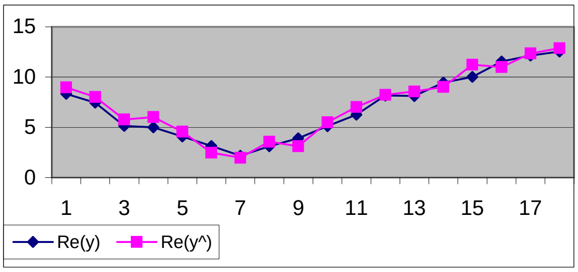

Fig. 5. Dependence of real parts of vectors of observation and errors of one-step-by-step forecasting on the ninth iteration

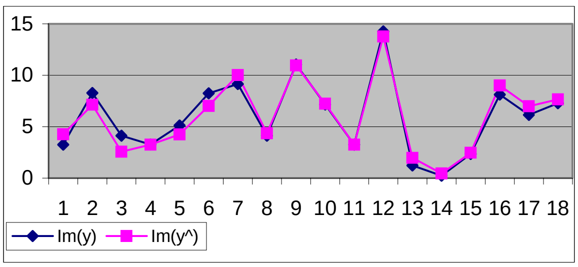

Fig. 6. Dependence of imaginary parts of vectors of observation and errors of one-step-by-step forecasting on the ninth iteration

The analysis of the given dependences shows, that in space of the responses we also managed to raise(increase) quality of forecasting, and

$$\frac{\sqrt{\left\|\tilde{Y}^9 - \hat{Y}^9\right\|}}{\sqrt{\left\|\tilde{Y}^0 - \hat{Y}^0\right\|}} * 100\% \approx 15\% \text{ in space of the responses.}$$

CONCLUSION

As a result of the carried out researches was developed and realized within the framework of program system MATLAB procedure of optimum parametrical estimation in frequency domain for stochastic linear discrete - time systems described by models in state space. For the first time was considered and solved in the most general statement the task of parametrical estimation, appropriate to a case, when estimated parameters are included in the managements, indignation, measurement, in matrices of state, control, noise, observation matrices and in the initial conditions and in the variance matrices of dynamic noise and errors of observation.

The given numerical results have allowed to make a conclusion about expediency of application and comparative efficiency of the frequency approach to the deciding the task of active identification of stochastic linear discrete systems.

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