SESSIONS 1

Natural Sciences

1.1. Mathematics

Efficient Algorithm of Low-Entropy Markov Source Coding

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Abstract-**The problem of coding low-entropy Markov sources is considered. Since the run-length code was offered about fifty years ago by Shannon [1], it is known that for such sources there exist coding methods much simpler than for sources of a general type. However the known coding methods of low-entropy sources does not give possibility to reach the given redundancy. On the based of code construction from [2], a new method of coding low-entropy Markov sources is offered. It permits to reach a given redundancy r with almost the same encoder and decoder memory size as for general methods while encoding and decoding are much faster.**

I. INTRODUCTION

The problem of coding low-entropy Markov source is considered. Elementary example of low-entropy source is a Bernoulli source generating a sequence of zeros and ones with probabilities q and p, respectively, when $p – 0$. The problem of low-entropy Bernoulli source coding has attracted attention of many researchers, as for coding of such sources there exist simpler methods than in a general case.

The efficiency of a code is measured by redundancy and by complexity of encoding and decoding. The redundancy r is a difference between the average codeword length and the Shannon entropy. Complexity is estimated by the memory size of the encoder and decoder (in bits) and by the average time of encoding and decoding one symbol measured by the number of binary operations on single-bit word when they are implemented on a computer with random access memory [3].

One of the well-known compression schemes of low-entropy Bernoulli sources is the run-length coding [1]. In [4] Elias proposes to use prefix code of integers for the run-length coding. An effective run-length coding method was offered by Golomb [5]. However the known methods of low-entropy sources [1, 4, 5] do not allow reaching the given redundancy. In [2], a new method of coding low-entropy Bernoulli sources is offered. It permits reaching a given redundancy r with almost the sa.me encoder and decoder memory size as for general methods, while encoding and decoding is much faster.

Here on the based of code construction from [2], a new efficient algorithm of coding low-entropy Markov source with known statistics is offered.

II. ALGORITHM OF CODING LOW-ENTROPY MARKOV SOURCES

Let a Markov source of the order n generating a sequence of zeros and ones with conditional probabilities $p(0|x)$ and $p(1|x)$ respectively, where $x \in A^n$, $A = \{0,1\}$, A^n is a set of words of the length n in A.. Let $r>0$ be the given redundancy of a code. Our problem is to construct a method of source coding permitting us to reach the given redundancy r.

In our method encoding is implemented in two stages: first, a message is compressed by a simple code and an output sequence is then encoded by a fast and effective code. After the first stage the length of the input sequence is essentially reduced, and applying rather a complex fast algorithm at the second stage provides little total time of encoding and decoding per letter of the initial message. At the second stage we shall use the code from [6].

For code from [6] for Bernoulli source the dependence of the memory size V and the average time T of encoding and decoding of one letter on the redundancy r' as $r' \to 0$ satisfies the following estimates:

$$V = O\left(\frac{1}{r'}\log\frac{1}{r'}\right),\ T = \left(\log^3\left(\frac{1}{r'}\right)\log\log\frac{1}{r'}\right).$$

Let $p(1|x)<p(0|x)$ for all $x\in A^n$, $A=\{0,1\}$. Let $x_1x_2...x_s...$ be a sequence generated a Markov source of the order n, $x_i\in A$. Consider the first stage of encoding. We divide the input message into blocks of the length $l = \left\lceil 1/\sqrt{p_{\max}} \right\rceil$, where

$$p_{\max} = \max_{x_1...x_n} p(1|x_1...x_n) \qquad (1).$$

If a block consists completely of zeros, then its code is zero. Otherwise we encode it according to the following rule: the first letter of the codeword is 1 followed by the same block the length l.

Let now $y_1y_2...y_t$ be a sequence obtained after the first stage of encoding, $y_i \in A$, $A=\{0,1\}$. Consider the second stage of encoding realized by a fast code from [6]. Note that the sequence $y_1y_2...y_t$ cannot already be considered as a Markov one, therefore we offer the following method. First, for convenience we present a sequence $y_1y_2...y_t$ as

$$\underline{0}\ \dots\ \underline{0}\ \underline{1}\ \underbrace{y_1...y_l}_{l}\ \underline{0}\ \dots\ \underline{0}\ \underline{1}\ \underbrace{y_1...y_l}_{l}\ \dots\ .$$

In this sequence we mark blocks of length l following after appearance of 1, and "special" symbols $\underline{0}$ and $\underline{1}$ not included in blocks. Encoding of various y_i is implemented with the help of various encoders "tuned" to various probabilities of appearance of zeros and ones and is in the following.

Let $x_1'...x_n'$ is a symbols sequence before block $\underbrace{0...0}_{l}$ of the initial sequence. Then the "special" symbols $\underline{0}$ and $\underline{1}$ areencoded with the help of an encoder K_0 with probabilities s and $(1 - s)$ respectively, where

$$s = p(0|x_1'...x_n')p(0|x_2'...x_n'0)...p(0|x_l'...x_n'\underbrace{0...0}_{l-1}).$$

Let us consider encoding of symbols inside the block $y_1...y_l$. Let $y_1...y_{i-1} = \underbrace{0...0}_{i-1}$ (i = 1,2,…l). Let the block $x_{n+1}...x_{n+l}$ of initial sequence refer to the block $y_1...y_l$. Then symbol y_i following $(i-1)$ zeros is encoded with the help of an encoder K_i with probabilities q_i, if $y_i = 1$, and $(1 - q_i)$, if $y_i = 0$, where

$$q_i = \frac{p(1|x_i...x_n\overbrace{0...0}^{i-1})}{1 - p(0|x_i...x_n\underbrace{0...0}_{i-1})...p(0|x_l...x_n\underbrace{0...0}_{l-1})}$$

The symbols following 1 in the block $y_1...y_l$ are encoded with the help of an encoder $\bar{K}$ with initial probabilities $p(0|x)$ and $p(1|x)$ for 0 and 1, respectively. Formally the algorithm of encoding can be described as follows:

Step 1. τ_i is calculated and y_i is encoded with the probability q_i, if $y_i = 1$, or $(1-q_i)$, if $y_i=0$.

Step 2. If $y_i = 1$, then all the symbols following y_i are encoded with initial conditional probabilities $p(0|x)$ and $p(1|x)$ for 0 and 1, respectively. Otherwise turn to the following symbols and return to Step 1.

Let $p(1|x)<p(0|x)$ not for all $x\in A^n$, $A=\{0,1\}$. We present the initial sequence according to the following rule. If $p(0|x)<1/2$ then all the zeros following after this $x\in A^n$ are encoded by 1. Otherwise all the zeros are encoded by 0. Analogously, if $p(1|x)<1/2$ then all the ones following after this $x\in A^n$ are encoded by 1. Otherwise all the ones are encoded by 0.

At the first stage the counters of zeros of the size $O\left(\log\frac{1}{p_{\max}}\right)$ are stored in memory. The given method of Markov source coding has the same memory of the encoder and decoder as method of Bernoulli source coding in [2] (it has the same memory as "general" methods) multiplying on 2^n, when n is the order of Markov source. The properties of proposed method are characterized the following theorem. Let there be given a Markov source of order n $(n>0)$ generating a sequence of zeros and ones with conditional probabilities $p(0|x)$, $p(1|x)$ $(x \in \{0,1\}^n)$, and $0<r<1$. Let the above-described code with $l = \left\lceil 1/\sqrt{p_{\max}} \right\rceil$ at the first stage, where p_{max} is determined by formula (1), and theredundancy $\bar{r} = r/2$, at the second stage be used. Then the general redundancy of the code does not exceed r, and the average time T of encoding and decoding of one symbol satisfies the following inequalities:

$$T < C_1\sqrt{p_{\max}}\log\frac{1}{rp_{\max}}\log\log\frac{1}{rp_{\max}}\log\log\log\frac{1}{rp_{\max}} + C_2,$$

where C_1, C_2 are constants.

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