Numerical determination of Frequency distribution function for 2d fokker–planck equation

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***Abstract* – In this paper a numerical determination of frequency distribution function for Fokker–Planck equation is considered. To do this, the new iterative method was constructed and applied to the parabolic equation with boundary conditions of first kind (at least one particle reaching the frontier of domain). The strength and powerful of proposed method are that the new factors as time dependence and fluctuation matrix took into account. These factors change the structure of numerical algorithm significantly. Algorithm is written in a matrix form. The theorem proving the convergence and stability of iterative process is put in.**

I. INTRODUCTION

When parabolic equations are being solved numerically, the fractional step method and method of weak approximation are being rarely used because derivative along the normal vector from unknown function on a border does not correspond to direction of flow vector. Hence, problems of adequate calculations of boundary conditions and constructing absolutely stable implicit finite difference scheme on each time step are arisen.

Nowadays to solve systems of linear algebraic equations with high order sparse matrix the iterative methods are applied, e.g. incomplete factorization implicit method (IMIF-53), conjunctive gradient squared method (CG-S), stabilized bi-conjunctive gradient method (Bi-CGSTAB) [1] and others. The most powerful of them are constructed by using the factorization technique, when the general operator Q of finite–differential scheme written in canonical form $Q\left(U^{s+1}-U^{s}\right)\cdot\tau_s^{-1}+MU^{s}=F$, where τ – time step, U^s – unknown function (solution of equation), presented as $Q\approx Q_1\circ Q_2\circ\ldots\circ Q_p$ or in more simple form.

It should be noted that the general advantages of such iterative processes are high practice economy and broad abilities to construct self-adapting algorithms for solving different boundary problems.

II. PROBLEM DEFINITION AND ITERATIVE ALGORITHM CONSTRUCTING

Let's consider Fokker–Planck boundary problem in 2D Cartesian domain $G=(0,a)\times(0,b)$:

$$p'_t=\partial/\partial x\left(k_{11}p'_x\right)+2\,\partial/\partial x\left(k_{12}p'_y\right)+\partial/\partial y\left(k_{22}p'_y\right), \quad (1)$$

$$p\big|_{\Gamma}=1,\ p\big|_{t=0}=\begin{cases}0, (x,y)\in \text{int } G\\ 1, (x,y)\in\Gamma,\end{cases} \quad (2)$$

where $k_{i,j}(x,y)$, $i=1,2;\quad j=1,2$, – elements of fluctuation matrix Λ, suited to parabolic criterion $k_{11}k_{22}-k_{12}^2\ge 0$; $\Gamma=\partial G$ – frontier of G, p – frequency distribution function (transitional probability), x – share price, y – quantity of shares in some share holding.

As you probably know, boundary problem (1) – (2) describe a behaviour of stochastic process $S(x,y,t)$ that is the price of some asset with transition probability $p(x,y,t)$. The function $S(x,y,t)$ satisfies to stochastic differential equation (in Ito meaning; to obtain more detailed information see, e.g., [2]):

$$d\,S=k_{11}\,S\,dW_1+k_{22}\,S\,dW_2+2k_{12}\,dt\ ,$$

where $k_{12}=corr\left(dW_1,dW_2\right)$, dW_i, $i=1,2,$ – one–dimension Wiener process, k_{11}, k_{22} – volatility (measure of risks).

After approximation procedure equation (1) can be written in finite – differential form as follows:

$$A_{i,n}U^k_{i-1,n-1}+B_{i,n}U^k_{i,n-1}+L_{i,n}U^k_{i+1,n-1}+K_{i,n}U^k_{i-1,n}-$$
$$-C_{i,n}U^k_{i,n}+E_{i,n}U^k_{i+1,n}+D_{i,n}U^k_{i-1,n+1}+V_{i,n}U^k_{i,n+1}+$$
$$+Y_{i,n}U^k_{i+1,n+1}+F^{k-1}_{i,n}=0, \quad (3)$$

$i=\overline{2,N-1}$, $n=\overline{2,P-1}$, $k=\overline{1,M_1}$, $A_{i,n}$, $B_{i,n}$, $L_{i,n}$, $K_{i,n}$, $C_{i,n}$, $E_{i,n}$, $D_{i,n}$, $V_{i,n}$, $Y_{i,n}$, $F_{i,n}$ – some coefficients,

k – time layer number, i, n – serial number of spatial variable layer along X and Y directions accordingly.

Let boundary conditions (2) approximated as

$$U_{1,n}^k = \varphi_{2,n}U_{2,n}^k + \xi_{2,n},\ U_{i,1}^k = \varphi_{i,2}U_{i,2}^k + \xi_{i,2},$$

$$U_{N,n}^k = \varphi_{N-1,n}U_{N-1,n}^k + \xi_{N-1,n},$$

$$U_{i,P}^k = \varphi_{i,P-1}U_{i,P-1}^k + \xi_{i,P-1}.$$

Factorization of differential scheme (3) with boundary conditions should be made by using special relations:

$$U_{i,n}^k = \alpha_{i+1,n}U_{i+1,n}^k + \beta_{i+1,n,k},\ U_{i,n}^k = \overline{\alpha}_{i,n+1}U_{i,n+1}^k + \overline{\beta}_{i,n+1,k};$$

$$U_{i,n}^k = \gamma_{i-1,n}U_{i-1,n}^k + d_{i-1,n,k},\ U_{i,n}^k = \overline{\gamma}_{i,n-1}U_{i,n-1}^k + \overline{d}_{i,n-1,k}.$$

These ones must be substituted into (3).

For example, the first expression of the iterative process is as follows:

$$\alpha_{i,n}^{(0)} = 0;\ \overline{\alpha}_{i,n}^{(0)} = 0;\ \gamma_{i,n}^{(0)} = 0;\ \overline{\gamma}_{i,n}^{(0)} = 0;\ i = \overline{1,N},\ n = \overline{1,P};$$

$$\alpha_{i+1,n}^{s+1/2} = (E_{i,n}, + L_{i,n}\cdot\overline{\alpha}_{i+1,n}^{s+1/2} + Y_{i,n}\cdot\overline{\gamma}_{i+1,n}^{s})M_s^{-1},$$

where $M_s = C_{i,n} - B_{i,n}\overline{\alpha}_{i,n}^{s+1/2} - V_{i,n}\overline{\gamma}_{i,n}^{s} -$

$$-\alpha_{i,n}^{s+1/2}\left(K_{i,n} + A_{i,n}\overline{\alpha}_{i-1,n}^{s+1/2}\right) - \alpha_{i,n}^{s+1/2}D_{i,n}\overline{\gamma}_{i-1,n}^{s};$$

$$\alpha_{2,n}^{s+1/2} = \varphi_{2,n},\ i = \overline{2,N-1},\ n = \overline{2,P-1}.$$

If $F_{i,n}^{k-1}$ doesn't depend on differential function $U_{i,n}^{k-1}$ (it means that (1) is an elliptic equation) and $k_{12} = 0$ then well-known formulas of iterative algorithm will be obtained [3].

To finish a construction of our iterative process (we will name it as "α–β" algorithm) the criterion of the convergence should be chosen. It is very important point because it will define appropriate rate of convergence (RC). The most famous criteria are:

1. $\left\| z^s \right\|_D \equiv \left\| u^s - u \right\|_D \le \varepsilon$, where $\| u \|_D = (Du, u)$, u – exact solution, s – number of iterations;
2. $\left\| u^s - u \right\|_D \le \varepsilon\cdot\left\| u^{(0)} - u \right\|_D$, $u^{(0)}$ – initial value, s – number of iterations.

These criteria have very good comments got from men who making computations in practice. Actually, if $D = M^*M$, where M – self-adjoint operator of differential equation (1), had written in common operational form $Mu = f$, then $\left\| z^s \right\|_D \equiv \left\| u^s - u \right\|_D =$

$$= \left(M\left(u^s - u\right), M\left(u^s - u\right)\right) = \left\| Mu^s - f \right\|^2 \equiv \| r_s \|^2,$$ where $r_s = \left(Mu^s - f\right)$ – residual, $z^s = \left(u^s - u\right)$ – error of solution. Obviously, $\left\| z^s \right\| \le \left\| M^{-1} \right\|\cdot\| r_s \|$ and any desired precision of iterative method can be obtained. Nevertheless, when operator norm $\left\| M^{-1} \right\|$ is unknown or it is estimated very rough then criteria of convergence are used as follows [3,4]:

$$\max_{i,n}\left|\left(U_{i,n}^{s+1} - U_{i,n}^{s}\right)\Big/U_{i,n}^{s}\right| < \varepsilon \text{ or } \left\| U^{s+1} - U^s \right\| < \varepsilon. \quad (4)$$

To prove its reliability let's consider some iterative algorithm written in a matrix form $U^s = Q\cdot U^{s-1} + g$. Since inequality $\left\| U^{s+1} - U^s \right\| \le \|Q\|\cdot\left\| U^s - U^{s-1} \right\|$ is executed or as the same $\left\| z^s \right\| \le \rho\left\| z^{s-1} \right\|$, where $\|Q\| = \rho$ – the convergence radius of iterative method, we can get that

$$\left\| U - U^s \right\| = \left\| U - U^s + U^{s+1} - U^{s+1} + \ldots + U^{s+k} - \right.$$

$$\left. - U^{s+k} + \ldots \right\| \le \left\| U^{s+1} - U^s \right\| + \ldots + \left\| U^{s+k+1} - U^{s+k} \right\| +$$

$$+ \ldots \le \left\| U^{s+1} - U^s \right\|\cdot\left(1 + \rho + \rho^2 + \ldots\right) \le \varepsilon(1-\rho)^{-1}.$$

Obviously, if iterative process converge slowly to exact solution (i.e. $\rho \approx 1$) then criteria (4) is not allowed to be used because in this case to obtain accuracy ε the method is to be iterated in $(1-\rho)^{-1}$ times more.

As our iterative algorithm is nonlinear and norm $\left\| M^{-1} \right\|$ is unknown the expression (4) is to be used.

III. PRACTICE COMPUTATIONS

In spite of lengthy of proposed "α–β" process it has high performance approved by practical computations. It doesn't require any information about differential operator spectrum. Also it has very high rate of convergence (RC) – it is much more then have the other methods.

Let us solve boundary problem (1) – (2) with parameters as follows:

$$G = (0,1)\times(0,1),\ k_{11} = 1 + c\left(x^2 + y^2 - 2x - 2y + 2\right),\ k_{12} = 0,$$

$k_{22} = 1 + c\left(1 - (x-1)^2 + (y-1)^2\right)$, $c > 0$ – any real number. Let we have one extra expression:

$$p_t' = 0.$$

The analytical solution (time independent) to the problem is

$$p(x,y) = \begin{cases} 1, (x,y) \in \overline{G} \\ 0, (x,y) \notin \overline{G}. \end{cases}$$

For boundary problem (1) – (2) coefficients of algebraic system (3) are as follows:

$$K_{i,n} = h_1^{-2} k_{11}\left(0.5\left(x_i + x_{i-1}\right), y_n\right),$$

$$E_{i,n} = h_1^{-2} k_{11}\left(0.5\left(x_i + x_{i+1}\right), y_n\right),$$

$$V_{i,n} = h_2^{-2} k_{22}\left(x_i, 0.5\left(y_n + y_{n+1}\right)\right),$$

$$C_{i,n} = K_{i,n} + E_{i,n} + B_{i,n} + V_{i,n},$$

$$B_{i,n} = h_2^{-2} k_{22}\left(x_i, 0.5\left(y_n + y_{n-1}\right)\right),$$

$$A_{i,n} = L_{i,n} = D_{i,n} = Y_{i,n} = 0,$$

where $h_i = N_i^{-1}$, $i = 1, 2$ – mesh step along axes X and Y respectively, N_i – number of steps with length h_i.

Practical computations showed that "α–β" algorithm converges to the analytical solution with RC equivalenting to $O\left(0.5 \cdot \ln^{-1} N\right)$. Incomplete factorization implicit method IMIF–53 has RC $O\left(N^{-2/3}\right)$. And, at last, conjunctive gradient squared method CG-S as well as stabilized bi-conjunctive gradient method Bi-CGSTAB are judged as the slowest processes in the group, because its rates are $O\left(N^{-6/7}\right)$ only, where $N = \max\left(N_1, N_2\right)$.

For problem (1) – (2) dynamics of convergences depended on iterations is shown in the fig.1.

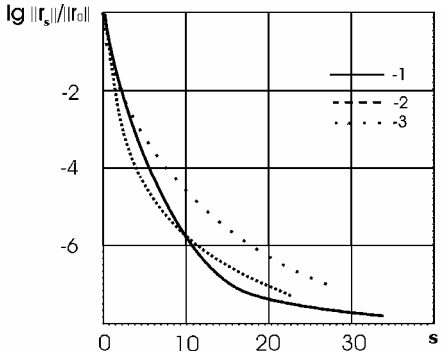

Fig. 1. Convergence dynamics of iterative methods
$\|r_0\| = 0$, $\|z^0\| = 6,4 \cdot 10^4$, $N = 64$
(1 – "α–β" algorithm, 2 – Bi-CGSTAB, 3 – IMIF–53
(by data from [1]))

The most important parameters of each mentioned algorithm wrote down in the table 1.

TABLE 1
PARAMETERS OF ITERATIVE METHODS

ALGRITHM	ARITHMETIC OPERATIONS	MEMORY ARRAYS	MESH, POINTS	TIME, S
BI-CGSTAB	$15N_1N_2M$	8	512×512	2,9
CG-S	$16N_1N_2M$	10	512×512	3,2
IMIF-53	$18N_1N_2M$	10	512×512	3,5
α–β	$78N_1N_2M$	8	512×512	2,4

In the table 1 parameter N_1 means the quantity of mesh points along axis X, N_2 – along axis Y and M – along time line. Parameter "time" counts a quantity of time needed to solving the problem (1) – (2).

The convergence of α – process (i.e. only one half of the whole iterative algorithm) applied to elliptic equation (i.e., (3) have coefficients $A_{i,n} = L_{i,n} = D_{i,n} = Y_{i,n} = 0$) is proved theoretically in [3].

But complete proof didn't obtain before this moment.

IV. THEORETICAL CONVERGENCE OF ALGORITHM

Let we have an uniform mesh, where $N = N_1 = N_2$. If we denote the matrix of the coefficients as $U_W^s = \left\{W_{i,j}^s,\ i = \overline{1, N},\ j = \overline{1, N}\right\}$, where W – some element of the family $\left\{\alpha, \beta, \bar{\alpha}, \bar{\beta}, \gamma, d, \bar{\gamma}, \bar{d}\right\}$, s – number of iteration, and define submatrix $\hat{U}_W^s = \left\{W_{i,j}^s, i = \overline{3, N},\ j = \overline{2, N-1}\right\}$, as well as $\breve{U}_W^s = \left\{W_{i,j}^s,\ i = \overline{1, N-2},\ j = \overline{2, N-1}\right\}$. Let central point matrix is $\bar{U}_W^s = \left\{W_{i,j}^s,\ i = \overline{2, N-1},\ j = \overline{2, N-1}\right\}$ and initial matrix is $U_0 = \left\{u_0(x_i, y_j),\ i = \overline{1, N},\ j = \overline{1, N}\right\}$.

The following lemma can be proved.

Lemma. "α–β" algorithm can be written in canonical form:

$$Q_s \tau_s^{-1}\left(\hat{U}_\alpha^{s+1} - \hat{U}_\alpha^{s+1/2}\right) + M_s \hat{U}_\alpha^{s+1/2} = F_s,$$

$$Q_s \tau_s^{-1}\left(\hat{U}_\beta^{s+1} - \hat{U}_\beta^{s+1/2}\right) + M_s \hat{U}_\beta^{s+1/2} = F_s,$$

with initial conditions

$$\hat{U}_\alpha^{(0)} = 0,\ \hat{U}_{\beta,k}^{(0)} = \begin{cases} U_0, & k = 1 \\ U_{\beta,k-1}, & k \geq 2 \end{cases},$$

where $Q_s = \left(C_{i,n} \cdot I - B_{i,n} \cdot \bar{U}_{\bar{\alpha}}^{s+1/2} - V_{i,n} \cdot \bar{U}_{\bar{\gamma}}^{s+1} - \right.$

$\left. - \bar{U}_\alpha^{s+1} \cdot K_{i,n} \cdot I - \bar{U}_\alpha^{s+1}\left(A_{i,n} \cdot \breve{U}_{\bar{\alpha}}^{s+1/2} + D_{i,n} \cdot \breve{U}_{\bar{\gamma}}^{s+1}\right)\right)$,

$M_s = V_{i,n}\left(\bar{U}_{\bar{\gamma}}^{s} - \bar{U}_{\bar{\gamma}}^{s+1}\right) + K_{i,n}\bar{U}_\alpha^{s+1/2} - K_{i,n}\bar{U}_\alpha^{s+1} I +$

$+ \bar{U}_\alpha^{s+1/2}\left(A_{i,n}\breve{U}_{\bar{\alpha}}^{s+1/2} + D_{i,n}\breve{U}_{\bar{\gamma}}^{s}\right) - A_{i,n}\bar{U}_\alpha^{s+1}\breve{U}_{\bar{\alpha}}^{s+1/2} -$

$- D_{i,n}\bar{U}_\alpha^{s+1}\breve{U}_{\bar{\gamma}}^{s+1}$, $F_s = Y_{i,n}\left(\hat{U}_{\bar{\gamma}}^{s+1} - \hat{U}_{\bar{\gamma}}^{s}\right)$, τ_s – iterative parameter, k– time layer.

Theorem. Let matrix of algebraic system (3) satisfies to the condition of diagonal transformation and $C_{i,n} \geq V_{i,n} + K_{i,n} + E_{i,n} + B_{i,n}$, where $B_{i,n} \geq 0$, $V_{i,n} \geq 0$, $C_{i,n} \geq 0$, $K_{i,n} \geq 0, E_{i,n} \geq 0$, Let for each $s \geq 0$ $\left|\alpha_{2,n}^{(s)}\right| \leq 1$,

$\left|\gamma_{N-1,n}^{(s)}\right| \le 1$, $\left|\overline{\alpha}_{i,2}^{(s+1/2)}\right| \le 1$, $\left|\overline{\gamma}_{i,N-1}^{(s)}\right| \le 1$, $\left|\varphi_{2,n}\right| \le 1$, $\left|\varphi_{N-1,n}\right| \le 1$, $\left|\varphi_{i,2}\right| \le 1$, $\left|\varphi_{i,N-1}\right| \le 1$, $\left|\xi_{2,n}\right| \le C_d$, $\left|\xi_{N-1,n}\right| \le C_d$, $\left|\xi_{i,2}\right| \le C_d$, $\left|\xi_{i,N-1}\right| \le C_d$, where $C_d > 0$ – some constant. Then "α–β" process converge and α–process will be stable.

IV. CONCLUSION

The modified "α–β" algorithm is constructed. It is applied to numerical solving of parabolic equation (1) with boundary conditions (2). Iterative method taking into account time dependence and dependence from fluctuation matrix. Algorithm is written in canonical matrix form and theorem for proving its convergence is formulated.

The practical computations approved high performance of proposed iterative process for parabolic equations (1)–(2) with high level of volatility and high variation of stochastic function $S(x,y,t)$ that is very important for making research in the field of financial mathematics [5].

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