Self Similar Spectrum of Fractal Lattice

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***Abstract*-Recursion relations for Green function of Sierpinski gasket vibrations are obtained. Dependences of oscillating two dimension lattice type and its spectrum were set.**

I. INTRODUCTION

As well known, there exists physical effect of localization in vibrating self similar lattices. On the high frequencies oscillations affects only a small part of it. Lowest frequency, where this effect occurs, called *crossover frequency*. This phenomenon is not studied well because of absence of effective analytic tools and high complexity of calculations needed to simulate.

My main goal is to discover if there is dependence, of topology and wholeness of oscillating grid and the shape of its spectrum. For this purpose complex of programs is written in MatLab environment, which calculates spectra and eigenfunctions for different fractal girds, particularly for Sierpinski carpet. Also recursion relations for Green function of Sierpinski gasket vibrations are obtained

II. MODEL

Model equation for wave amplitude has the form

$$\ddot{u}_i = \sum_j K_{ij}(u_j - u_i) \qquad (1)$$

K_{ij} = 1, if binding between junctions exists, and 0 in other case.

Equation would have physical meaning if we will suppose that there are vertically oscillating masses m in junctions of a grid.

We look for a solution in form $\vec{\alpha}e^{i\omega t}$

Equation will take form

$$-\omega^2 e^{i\omega t} = A\vec{\alpha}e^{i\omega t} \qquad (2)$$

$-\omega^2$ are eigenvalues of matrix **A**.

The task reduces to the eigenvalues search problem.

III. RECURSION FOR GREEN FUNCTION

According to William A. Shwalm matrix A from equation (2) is called Hamiltonian. Elements of matrix $g = [\omega^2 I + A]^{-1}$ are called Green functions.

Having Green functions $a_1, a_2, ..., a_n$ of some lattices we can easily obtain Green function of their interconnection. Let

$$A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} + B$$

Where matrix B realizing lattice interconnection. If $a_1, a_2, ..., a_n$ are Hamiltonians for their lattices, then A would be Hamiltonian for their interconnection. Simpler we can write $A = a + B$, substituting this into Green function we'll get

$$G = g + gBG$$

For Sierpinski gasket we use this scheme of concatenation.

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

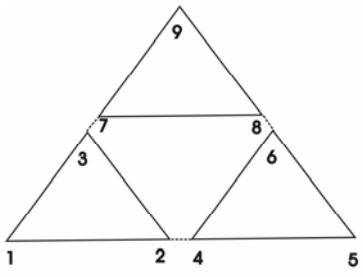

Fig. 1. Scheme of concatenation of three Sierpinski gaskets of n'th iteration to form n+1iteration gasket, and its sites numeration

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Let's count recursion equation for $G_{1,1}$. With elementary manipulations we obtain

$$G_{1,1} = g_{1,1} + g_{2,1}(G_{4,1} - G_{2,1})$$
$$G_{4,1} = g_{4,4}(G_{4,1} - G_{2,1})$$
$$G_{2,1} = g_{2,1} + (g_{2,2} - g_{2,3)}(G_{4,1} - G_{2,1})$$
$$G_{5,1} = g_{5,4}(G_{4,1} - G_{2,1})$$

Recursion involves only two distinct Green functions on each generation. Following William A. Schwalm we'll call them corner-to-corner Green function and corner-to-other-corner Green function. And denote them $X = G_{1,1} = \ldots, Y = G_{5,1} = \ldots$ accordingly.

Finally we obtain recursion relations

$$X = x + \frac{2y^2}{1-2x+y}, \quad Y = \frac{y^2}{1-2x+y},$$

where X, Y Green functions on the n+1 iteration and x, y Green functions on the n iteration.

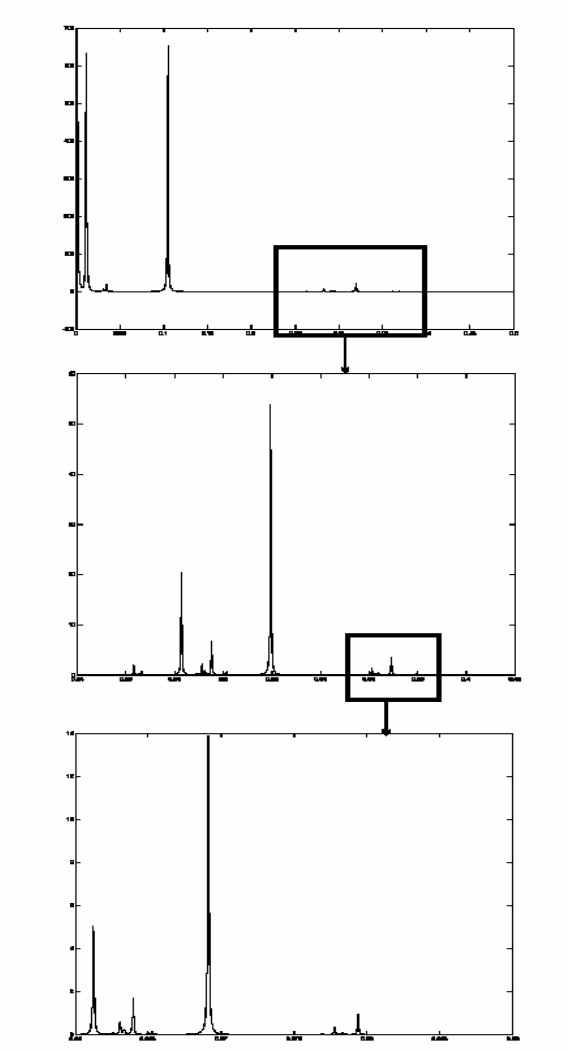

Fig. 2. Self similarity of Sierpinski gasket eiganvalues density

Starting from one point and Green function $x = y = \frac{1}{z}$, we can count Green function on any iteration.

According to [2] local density of states at site i is defined by

$$D_i(E) = \sum_{\nu} |u_{i,\nu}| \delta(E - \alpha_\nu)$$

where $\{u_\nu\}, and\ \{\alpha_\nu\}$ are sets of eigenvectors and eigenvalues correspondently.

Since we know Green function Reference [3] gives us way to approximate LDOS

$$D_i(E) = -\frac{1}{\pi} \operatorname{Im} G_{i,i}(E - i\eta)$$

Using approximate relation for $D_i(E)$ and recursion relations obtained we can easily explore shape of eigenvalues density.

Here is (Fig. 5) LDOS for the corner site at fiftieth iteration for the shortening ranges of E.

It is easy to see, that eigenvalues density has self-similar structure.

These results are in accord with results of computer experiments on fractal lattices, like Sierpinski carpet.

IV. EIGENVALUES DENSITY ON DIFFERENT LATTICES

For purpose of holding experiments with different lattices complex of programs in MatLab environment is written.

Computations were carried on regular rectangle lattice, lattice with randomly removed sites, Sierpinski carpet and its different variations, including multifracatal lattices with Sierpinski carpet in its base and random analog of Sierpinski carpet with template randomly chosen on each step of its construction.

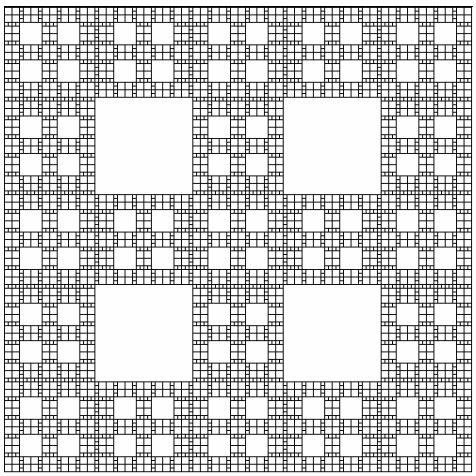

Fig. 3. Sierpinski carpet

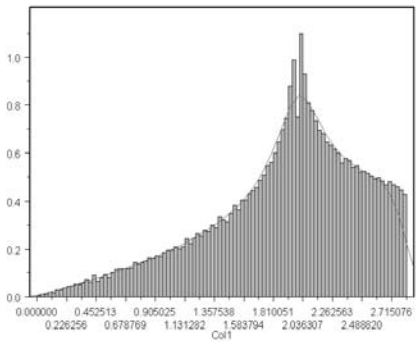

Fig. 3. Density of states for regular rectangle lattice

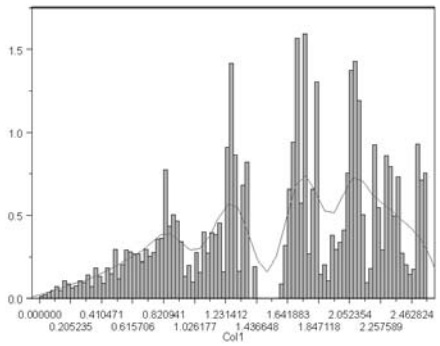

Fig. 4. Density of states for Sierpinski carpet

For the regular rectangle lattice density of states is smooth

In the case of Sierpinski carpet and its different variations density of states is quite irregular

In case of rectangle lattice with randomly removed sites density of eigenvalues is smooth and looks similar to regular rectangle gasket spectrum.

V. CONCLUSION

Investigated model allows catching of physical effect of localization. There were set some dependences between grid's type and it's spectrum, so that looking at spectrum we can understand if the vibrating lattice is self-similar in any mean or not.

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